

AN ARITHMETIC TRANSFER IDENTITY

ANDREAS MIHATSCH

1. INTRODUCTION

In [13], W. Zhang introduces his so-called Arithmetic Fundamental Lemma (AFL). This is a conjectural identity between certain derivatives of orbital integrals and certain intersection products in unitary Rapoport-Zink spaces, see [13, Conjecture 2.9]. Up to now, this conjecture has only been verified in the low dimensional cases $n = 2, 3$ by Zhang himself and in arbitrary dimension, but under restrictive conditions, in [8]. The method of proof is always an explicit computation of both sides of the identity.

In the present work we restrict to the case $n = 2$ to formulate and verify a variation of the AFL. The idea for this variant is due to W. Zhang and was communicated to us by M. Rapoport. We now explain our main results in detail. We will elaborate on the following definitions later in the paper.

Throughout this work we fix a prime $p \neq 2$ and a quadratic extension E/F of p -adic fields. We denote their rings of integers by $\mathcal{O}_F \subset \mathcal{O}_E$ and fix uniformizers π_F and π_E . Let \mathbb{F}_q be the residue field of F . Denote the non-trivial automorphism of E/F by $\sigma : a \mapsto \bar{a}$ and let η be the quadratic character of F^\times associated to E/F by class field theory. For any positive integer s , we denote by $\mathcal{O}_s := \mathcal{O}_F + \pi_F^s \mathcal{O}_E$ the order of conductor s .

We embed E^\times into $GL_2(E)$ via $x \mapsto \text{diag}(x, 1)$. In this way, E^\times acts by conjugation on $GL_2(E)$. An element $\gamma \in GL_2(E)$ is called regular semi-simple if its stabilizer under this action is trivial and if its orbit is Zariski closed.

Let $S(F) := \{\gamma \in GL_2(E) \mid \gamma\bar{\gamma} = 1\}$ and let $S(F)_{\text{rs}}$ denote its regular semi-simple elements. The symmetric space $S(F)$ is stable under conjugation by F^\times . We normalize the Haar measure on F^\times such that $\text{Vol}(\mathcal{O}_F^\times) = 1$. For $\gamma \in S(F)_{\text{rs}}$, for $f \in C_c^\infty(S(F))$ and for $s \in \mathbb{C}$, we define the following integrals:

$$\begin{aligned} \text{Orb}_\gamma(f, s) &:= \int_{F^\times} f(h^{-1}\gamma h) \eta(h) |h|^s dh, \\ \text{Orb}_\gamma(f) &:= \text{Orb}_\gamma(f, 0), \\ \partial \text{Orb}_\gamma(f) &:= \left. \frac{d}{ds} \right|_{s=0} \text{Orb}_\gamma(f, s). \end{aligned}$$

These integrals are absolutely convergent since, for regular semi-simple γ , the intersection $F^\times \gamma \cap \text{Supp}(f)$ is again compact.

Let $U(1)$ be the unitary group for a one-dimensional hermitian space for E/F . In particular, $U(1)(F) = \{x \in E \mid N_{E|F}(x) = 1\}$. We define the two unitary groups $U_0 := U(1 \oplus 1)$ and $U_1 := U(\varepsilon \oplus 1)$ where $\varepsilon \in F^\times$ is not a norm. The groups $U_0(F)$ and $U_1(F)$ are subgroups of $GL_2(E)$, stable under conjugation by $U(1)(F)$. We denote their regular semi-simple elements by $U_0(F)_{\text{rs}}$ and $U_1(F)_{\text{rs}}$ respectively. For $i \in \{0, 1\}$, for $\delta \in U_i(F)_{\text{rs}}$ and for $\varphi \in C_c^\infty(U_i(F))$, we define

$$\text{Orb}_\delta(\varphi) = \int_{U(1)(F)} \varphi(h^{-1}\delta h) dh.$$

Here the Haar measure is normalized such that $U(1)(F)$ has volume 1.

Two elements $\gamma \in S(F)_{\text{rs}}$ and $\delta \in U_i(F)_{\text{rs}}$ are said to match if they are conjugate under E^\times . By [13, Lemma 2.3], this relation defines a bijection of regular semi-simple orbits (of the actions of F^\times and $U(1)(F)$):

$$[S(F)_{\text{rs}}] \cong [U_0(F)_{\text{rs}}] \sqcup [U_1(F)_{\text{rs}}].$$

A function $f \in C_c^\infty(S(F))$ and a pair of functions $(\varphi_0, \varphi_1) \in C_c^\infty(U_0(F)) \times C_c^\infty(U_1(F))$ are said to be transfers of each other if for all $\gamma \in S(F)_{\text{rs}}$,

$$\omega(\gamma) \text{Orb}_\gamma(f) = \begin{cases} \text{Orb}_\delta(\varphi_0) & \text{if } \gamma \text{ matches } \delta \in U_0(F) \\ \text{Orb}_\delta(\varphi_1) & \text{if } \gamma \text{ matches } \delta \in U_1(F). \end{cases}$$

The transfer factor $\omega(\gamma) \in \mathbb{C}^\times$ will be defined in Section 3.

Now let \check{E}/\check{F} be the completions of the maximal unramified extensions of E and F with rings of integers $\mathcal{O}_{\check{F}} \subset \mathcal{O}_{\check{E}}$. Let \mathbb{F} be their residue field and \mathbb{X}/\mathbb{F} the unique formal p -divisible \mathcal{O}_F -module of height 2 and dimension 1. Let D be the quaternion algebra over F with ring of integers \mathcal{O}_D and standard involution $\iota : a \mapsto a^*$. We fix an isomorphism $\mathcal{O}_D \cong \text{End}(\mathbb{X})$ and endow \mathbb{X} with an \mathcal{O}_F -linear principal polarization such that the Rosati involution induces the standard involution on \mathcal{O}_D .

In addition, we endow \mathbb{X} with an action of \mathcal{O}_E such that an element $a \in \mathcal{O}_E$ acts on $\text{Lie}(\mathbb{X})$ via the structure morphism $\mathcal{O}_E \rightarrow \mathbb{F}$. This action is induced by an embedding $\mathcal{O}_E \subset \mathcal{O}_D$. We define $\bar{\mathbb{X}}$ to be the same p -divisible group (with the same \mathcal{O}_D -action), but with the σ -conjugated action of \mathcal{O}_E .

Now let X_i and Y_j be two quasi-canonical lifts of \mathbb{X} of levels i and j , defined over a finite extension $A/\mathcal{O}_{\check{F}}$ of ramification index e . We refer to [12] for the definition and properties of such lifts. We define

$$\mathbb{X}^{(2)} := \begin{cases} \mathbb{X} \times \bar{\mathbb{X}} & \text{if } E/F \text{ ramified or if } i+j \text{ is even} \\ \mathbb{X} \times \mathbb{X} & \text{if } E/F \text{ unramified and if } i+j \text{ is odd.} \end{cases}$$

The group $\mathbb{X}^{(2)}$ is endowed with the diagonal action of \mathcal{O}_E and the diagonal polarization.

Denote by $G \subset \text{Aut}_{\mathcal{O}_E}^0(\mathbb{X}^{(2)})$ the group of \mathcal{O}_E -linear quasi-isogenies which preserve the polarization. Then $G \subset GL_2(D)$ is a unitary group and there is an isomorphism

$$G \cong \begin{cases} U_1(F) & \text{if } E/F \text{ ramified or if } i+j \text{ is even} \\ U_0(F) & \text{if } E/F \text{ unramified and if } i+j \text{ is odd,} \end{cases}$$

as explained at the beginning of Section 4. We say that $\gamma \in S(F)_{\text{rs}}$ matches $g \in G$, if it matches an element in the correct unitary group which maps to g under this isomorphism.

For any $g \in G$, let $\text{Int}(g)$ be the $\mathcal{O}_{\check{F}}$ -length of the maximal closed subscheme of $\text{Spf}(A)$ to which g deforms as automorphism of $X_i \times Y_j$.

We define $K_{i,j} := \text{Stab}(\mathcal{O}_i \oplus \mathcal{O}_j) \cap U_0(F)$ if E/F is ramified or if $i+j$ is even. Otherwise we set $K_{i,j} := \text{Stab}(\mathcal{O}_i \oplus \mathcal{O}_j) \cap U_1(F)$. In either case, $1_{K_{i,j}}$ is the characteristic function of $K_{i,j}$. Then our main results are the following two theorems.

Theorem 1.1. *There exists a function $f \in C_c^\infty(S(F))$ which is a transfer of*

$$\begin{cases} (e \cdot 1_{K_{i,j}}, 0) & \text{if } E/F \text{ ramified or if } i+j \text{ is even} \\ (0, e \cdot 1_{K_{i,j}}) & \text{if } E/F \text{ unramified and if } i+j \text{ is odd} \end{cases}$$

with the following property. For any $\gamma \in S(F)_{\text{rs}}$ matching g in G , the length $\text{Int}(g)$ is finite and there is an equality

$$\omega(\gamma) \partial \text{Orb}_\gamma(f) = \text{Int}(g) \cdot \log q. \quad (1.1)$$

Theorem 1.2. (*Arithmetic transfer identity*) For any function $f \in C_c^\infty(S(F))$ which is a transfer of

$$\begin{cases} (e \cdot 1_{K_{i,j}}, 0) & \text{if } E/F \text{ ramified or if } i+j \text{ is even} \\ (0, e \cdot 1_{K_{i,j}}) & \text{if } E/F \text{ unramified and if } i+j \text{ is odd,} \end{cases}$$

there exists a function $f_{\text{corr}} \in C_c^\infty(S(F))$ with the following property. For any $\gamma \in S(F)_{\text{rs}}$ matching g in G , the length $\text{Int}(g)$ is finite and there is an equality

$$\omega(\gamma) \partial \text{Orb}_\gamma(f) = [\text{Int}(g) + \omega(\gamma) \text{Orb}_\gamma(f_{\text{corr}})] \cdot \log q. \quad (1.2)$$

These results can be considered as part of the program initiated by W. Zhang whose aim is to extend the range of the applicability of the AFL conjecture. The AFL conjecture is formulated for an unramified extension E/F and “trivial level structure” $i = j = 0$. The involved Rapoport-Zink spaces are formally smooth and the geometric side of the AFL is defined via intersection theory. Furthermore, the function on $S(F)$ whose orbital integrals are supposed to express the arithmetic intersection products in question is explicitly given and of a very simple nature. Also its transfer to the unitary side is explicitly given.

By contrast, our extension E/F is possibly ramified and $i, j \geq 0$. Then the function from Theorem 1.1 is no longer explicit. Although it is possible to write down such a function f in coordinates, there is neither a natural nor a convenient choice. Note that even if $i = j = 0$, we do not know a natural choice for f in the ramified situation.

Note that there is a list of moduli problems where one can conjecture an arithmetic transfer identity, see [7]. In all these cases, the involved Rapoport-Zink spaces are regular. Our problem does not appear on this list since for $i, j > 0$, there appear non-regular moduli spaces in the definition of $\text{Int}(g)$. This is why we chose to give an ad hoc definition of $\text{Int}(g)$ as a length.

There are two ingredients in the proof of the arithmetic transfer identity. The first is the formula of Gross and Keating for the deformation lengths of quasi-endomorphisms of quasi-canonical lifts, see [2], [3] and [4]. See also the account given in [1]. We need a slight extension of this formula as presented in [5, Section 7]. The second ingredient comes from harmonic analysis. More precisely, one has to know the existence of various test functions on the symmetric space $S(F)$. The case of general n was solved by W. Zhang by reduction to the Lie algebra, see [15]. Here we give direct proofs in the case $n = 2$.

The layout of this paper is as follows. In Section 2 we present a computation of the quantities $\text{Int}(g)$ following [5, Section 7]. This determines the right hand side of (1.1). In Section 3 we give a complete characterization of functions of the form $\gamma \mapsto \text{Orb}_\gamma(f)$ and $\gamma \mapsto \partial \text{Orb}_\gamma(f)$. In Section 4 we prove the theorems stated above.

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2. DEFORMATION OF HOMOMORPHISMS OF QUASI-CANONICAL LIFTS

In this section we first compute the space of homomorphisms between two quasi-canonical lifts. Then we derive an explicit formula for the length of the deformation locus of a quasi-homomorphism between quasi-canonical lifts. Our computations are essentially done in the work of S. Kudla and M. Rapoport, see [5, Section 7]. There is a mistake in their Lemma 7.4 which we correct in Lemma 2.3.

In this section we do not exclude the case $p = 2$. Apart from that, we use the notation from the introduction.

Let M_s/\check{E} be the ring class field associated to the order \mathcal{O}_s , with ring of integers W_s and maximal ideal \mathfrak{m}_s . If $s \geq 1$, then its ramification index over \check{F} is

$$e_s := [\mathcal{O}_E^\times : \mathcal{O}_s^\times] = \begin{cases} 2q^s & \text{if } E/F \text{ is ramified} \\ q^s + q^{s-1} & \text{if } E/F \text{ is unramified.} \end{cases}$$

If $s = 0$, then $M_s = \check{E}$.

2.1. Quasi-canonical lifts. Let X_0 be the Lubin-Tate module associated to the series

$$[\pi_E]_0(t) := \begin{cases} \pi_E t + t^q & \text{if } E/F \text{ is ramified} \\ \pi_E t + t^{q^2} & \text{if } E/F \text{ is unramified.} \end{cases}$$

This is a formal \mathcal{O}_E -module over \mathcal{O}_E in the sense of [10]. But we consider it as a formal \mathcal{O}_F -module over $\mathcal{O}_{\check{E}}$. Let \mathbb{X} be its reduction modulo π_E , which is a formal \mathcal{O}_F -module of height 2 over \mathbb{F} . The last condition means that the multiplication by any uniformizer of \mathcal{O}_F has height 2. The module X_0 will be called the canonical lift of \mathbb{X} .

By [12, Theorem 1.1], we can fix an isomorphism $\text{End}(\mathbb{X}) \cong \mathcal{O}_D$. The \mathcal{O}_E -action on X_0 induces an embedding $\mathcal{O}_E \hookrightarrow \mathcal{O}_D$ such that the action of \mathcal{O}_E on $\text{Lie}(\mathbb{X})$ agrees with $\mathcal{O}_E \rightarrow \mathbb{F}$. Composing this action with σ yields $\overline{\mathbb{X}}$ and $\overline{X_0}$ which are formal \mathcal{O}_F -modules with \mathcal{O}_E -action.

Let $\Pi \in \mathcal{O}_D$ be the Frobenius $t \mapsto t^q$. If E/F is ramified, then $\Pi = \pi_E$. If E/F is unramified, then $\Pi a = \bar{a}\Pi$ for all $a \in \mathcal{O}_E$ and $\Pi^2 = \pi_E$. In either case, Π uniformizes \mathcal{O}_D .

The following facts are proven in [12]. Let T be the p -adic Tate-module of the generic fiber of X_0 . It is a free \mathcal{O}_E -module of rank one generated by t , say. Any \mathcal{O}_F -superlattice $T \subset S \subset T \otimes_{\mathcal{O}_E} E$ defines, after a finite extension $A/\mathcal{O}_{\check{E}}$, a finite subgroup $S/T \subset X_0$. The quotient $X_0 \rightarrow X$ is a formal \mathcal{O}_F -module over A . We call S minimal of level s , if $S = (\mathcal{O}_E + \pi_F^{-s} \mathcal{O}_F)a \cdot t$ for some $a \in \mathcal{O}_E^\times$.

Let S be minimal of level s with corresponding quotient $\alpha_s : X_0 \rightarrow X'_s$. This quotient is defined over W_s . The isogeny α_s reduces to Π^s on the special fibre and so X'_s is a deformation of \mathbb{X} . The endomorphisms of X'_s can be described in two ways. The first is to consider $\text{End}(X'_s)$ as a subset of $\text{End}(\mathbb{X}) = \mathcal{O}_D$ by the reduction of homomorphisms. This subset coincides with $\mathcal{O}_s \subset \mathcal{O}_E$. The second is as explained in [12, Corollary 2.3]. Here $\mathcal{O}_s \subset \text{End}(X_0) = \mathcal{O}_E$ are the elements ϕ such that there exists ϕ' with $\alpha_s \circ \phi = \phi' \circ \alpha_s$. Again this induces an isomorphism $\mathcal{O}_s \cong \text{End}(X'_s)$.

These two actions of \mathcal{O}_s on X'_s coincide if E/F is ramified or if s is even. Otherwise they differ by the Galois conjugation of E/F . We define X_s to be the formal \mathcal{O}_F -module X'_s together with the first \mathcal{O}_s -action. This means that $a \in \mathcal{O}_s$ acts on $\text{Lie}(X_s)$ via $\mathcal{O}_s \subset W_s$. The formal module X_s is called a quasi-canonical lift of level s . The set of isomorphism classes of quasi-canonical lifts of level s is a principal homogeneous space under the group $\mathcal{O}_E^\times/\mathcal{O}_s^\times$ via its action on the minimal lattices.

Finally, if $A/\mathcal{O}_{\tilde{E}}$ is a finite extension and X/A a deformation of \mathbb{X} with $\text{End}(X) = \mathcal{O}_s \subset \text{End}(\mathbb{X})$, then $W_s \subset A$ and X is a quasi-canonical lift of level s .

Proposition 2.1. *Let X_i and Y_j be quasi-canonical lifts of level i and j defined over a finite extension $A/\mathcal{O}_{\tilde{E}}$. Then there exists an $a \in \mathcal{O}_E^\times$ and an equality*

$$\text{Hom}_{\mathcal{O}_F}(X_i, Y_j) = \Pi^{|i-j|} \mathcal{O}_{\min\{i,j\}} \cdot a.$$

Here the left hand side embeds into $\text{End}(\mathbb{X}) \cong \mathcal{O}_D$ by the reduction of homomorphisms.

Proof. The quasi-canonical lifts X_i and Y_j are defined by minimal superlattices of T as explained above. They take the form

$$S_i = (\pi^{-i} \mathcal{O}_F + \mathcal{O}_E) a_i \cdot t \quad \text{and} \quad S_j = (\pi^{-j} \mathcal{O}_F + \mathcal{O}_E) a_j \cdot t,$$

where $a_i, a_j \in \mathcal{O}_E^\times$ are chosen suitably.

Let $\alpha_i : X_0 \rightarrow X_i$ and $\alpha_j : X_0 \rightarrow Y_j$ be the corresponding isogenies. According to [12, Corollary 2.3], we have

$$\alpha_j^{-1} \circ \text{Hom}_{\mathcal{O}_F}(X_i, Y_j) \circ \alpha_i = \{x \in \mathcal{O}_E \mid xS_i \subset S_j\}.$$

With $\tilde{a} = a_i/a_j$, we get

$$\{x \in \mathcal{O}_E \mid xS_i \subset S_j\} = \begin{cases} \mathcal{O}_{\min\{i,j\}} \tilde{a} & \text{if } i < j \\ \pi_F^{i-j} \mathcal{O}_{\min\{i,j\}} \tilde{a} & \text{if } i \geq j. \end{cases}$$

Multiplying with Π^{-i} on the right and Π^j on the left yields the result with $a = \sigma^i(\tilde{a})$. \square

Corollary 2.2. *Let X_i and Y_j be quasi-canonical lifts of level i and j . Then*

- a) $\text{Hom}_{\mathcal{O}_F}(X_i, Y_j)$ is a free $\mathcal{O}_{\min\{i,j\}}$ -module of rank 1 where scalar multiplication is given by composition.
- b) The reduction of a homomorphism $X_i \rightarrow Y_j$ commutes with the \mathcal{O}_E -action on \mathbb{X} if and only if E/F is ramified or if $i+j$ is even. Otherwise it Galois-commute with the \mathcal{O}_E -action. \square

Recall [12, Proposition 4.6], stating that the modulus of X_s uniformizes W_s if $s \geq 1$. This means the following. After the choice of a formal coordinate $X_s \cong \text{Spf } W_s[[t]]$, we can write the multiplication by π_F as a power series

$$[\pi_F]_s(t) = \pi_F t + \dots + u_s t^q + \dots$$

Then u_s uniformizes W_s . In particular, $[\pi_F]_s$ has height 1 modulo \mathfrak{m}_s^2 . This implies the next lemma.

Lemma 2.3. *Consider two quasi-canonical lifts X_i and Y_j of levels $i > j$ defined over a finite extension A/W_i of ramification index e . Let $\mathfrak{m} \subset A$ be the maximal ideal. Then any automorphism of \mathbb{X} lifts to an isomorphism $X_i \otimes A/\mathfrak{m}^e \cong Y_j \otimes A/\mathfrak{m}^e$. But $X_i \otimes A/\mathfrak{m}^{e+1}$ and $X_j \otimes A/\mathfrak{m}^{e+1}$ are not isomorphic as \mathcal{O}_F -modules.*

Proof. According to [10, Theorem 3.8], the universal deformation space of \mathbb{X} is $\text{Spf } \mathcal{O}_{\tilde{E}}[[x]]$. Let $\varphi_i, \varphi_j : \mathcal{O}_{\tilde{E}}[[x]] \rightarrow A$ be the homomorphisms corresponding to X_i and Y_j . According to [10, Lemma 3.5], $\varphi_i(x) = u_i$ and $\varphi_j(x) = u_j$. Now $u_i \equiv u_j$ modulo \mathfrak{m}^e , but not modulo \mathfrak{m}^{e+1} . It follows that $X_i \otimes A/\mathfrak{m}^e \cong Y_j \otimes A/\mathfrak{m}^e$ and $X_i \otimes A/\mathfrak{m}^{e+1} \not\cong X_s \otimes A/\mathfrak{m}^{e+1}$.

We still have to show that any automorphism α of \mathbb{X} lifts to $X_i \otimes A/\mathfrak{m}^e$. But α induces an automorphism α^* of the universal deformation space and $(\varphi_i \circ \alpha^*)(x) \equiv \varphi_i(x) \pmod{\mathfrak{m}^e}$. \square

2.2. Deformation of homomorphisms. For the rest of this section we fix two quasi-canonical lifts X_i and Y_j of levels i and j defined over some finite extension $A/\mathcal{O}_{\tilde{F}}$. In particular $W_i, W_j \subset A$ and we let e be the ramification index of A over $W_{\max\{i,j\}}$. Let $\mathfrak{m} \subset A$ be the maximal ideal and set $d = |j - i|$.

We now compute the spaces $H_n := \text{Hom}_{\mathcal{O}_F}(X_i \otimes A/\mathfrak{m}^{n+1}, Y_j \otimes A/\mathfrak{m}^{n+1})$. We also abbreviate $H_\infty := \text{Hom}_{\mathcal{O}_F}(X_i, Y_j)$. Let $V_l \subset H_0 = \mathcal{O}_D$ denote the set of homomorphisms of height $\geq l$ and define $a(n) = 1 + q + \dots + q^n$.

Theorem 2.4. *Let $f_0 \in (H_\infty + V_l) \setminus (H_\infty + V_{l+1})$ and define $n := \lfloor (l+d)/2 \rfloor$. Then f_0 lifts to H_α but not to $H_{\alpha+1}$ with $\alpha + 1 =$*

$$e \cdot \begin{cases} a(l) & \text{if } l < d \\ a(n) + a(n-1) - a(d-1) & \text{if } d \leq l \leq i+j-1 \text{ and } l+d \text{ even} \\ 2a(n) - a(d-1) & \text{if } d \leq l \leq i+j-1 \text{ and } l+d \text{ odd} \\ 2a(j-1) - a(d-1) + \frac{l-(i+j-1)}{2} \cdot e_{\max\{i,j\}} & \text{if } i+j \leq l. \end{cases}$$

Remark 2.5. The fraction appearing in the theorem is always an integer. Namely $e_{\max\{i,j\}}$ is even except for the case $i = j = 0$ and E/F unramified. But if E/F is unramified, then the integer $l - (i + j - 1)$ in the last case is always even.

Proof. First we reduce to the case $i \leq j$. The formal group \mathbb{X} and the quasi-canonical lifts are also p -divisible groups. It is well known that the dual p -divisible \mathcal{O}_F -module \mathbb{X}^\vee is isomorphic to \mathbb{X} . (Here $\mathbb{X}^\vee = \text{Hom}(\mathbb{X}, \mathbb{L})$ where \mathbb{L} is a Lubin-Tate module for F .) It follows that X_i^\vee and Y_j^\vee are formal groups deforming \mathbb{X} with endomorphism ring equal to \mathcal{O}_i and \mathcal{O}_j . In particular, X_i^\vee and X_j^\vee are again quasi-canonical lifts of level i and j .

Now dualizing yields a bijection of \mathcal{O}_F -modules,

$$\text{Hom}(X_i \otimes W/\mathfrak{m}^{n+1}, Y_j \otimes W/\mathfrak{m}^{n+1}) \cong \text{Hom}(Y_j^\vee \otimes W/\mathfrak{m}^{n+1}, X_i^\vee \otimes W/\mathfrak{m}^{n+1}).$$

Its inverse is also given by dualization. This bijection commutes with the reduction of morphisms and so preserves the deformation lengths of homomorphisms. It also preserves the height and, in particular, preserves the spaces $(H_\infty + V_l) \setminus (H_\infty + V_{l+1})$. Here we defined H_n, H_∞ and V_l in the obvious way for the right hand side. So we can assume that $i \leq j$ from now on.

We can write $f_0 = h + g_0$ with $h \in H_\infty$ and $g_0 \in V_l$. It is clear that the deformations of f_0 and g_0 are in bijection via addition or subtraction of h .

So we can assume that f_0 has height l . Let X_i be defined by the lattice $(\mathcal{O}_E + \pi^{-i}\mathcal{O}_F)a_i \cdot t$ and define Y_j' by the lattice $(\mathcal{O}_E + \pi^{-j}\mathcal{O}_F)a_i \cdot t$. By Proposition 2.1, there exists an $a \in \mathcal{O}_E^\times$ which lifts to an isomorphism $Y_j \cong Y_j'$. Then left multiplication with a induces bijections $H_\infty \cong \text{Hom}(X_i, Y_j')$ and $H_n \cong \text{Hom}_{\mathcal{O}_F}(X_i \otimes A/\mathfrak{m}^{n+1}, Y_j' \otimes A/\mathfrak{m}^{n+1})$ and preserves the height. So it is enough to prove the theorem for $Y_j = Y_j'$.

For $i \leq k \leq j$, we define Z_k to be the quasi-canonical lift associated to $(\mathcal{O}_E + \pi^{-k}\mathcal{O}_F)a_i \cdot t$. For any $g_0 \in \mathcal{O}_D$ we define $n_k(g_0)$ to be the maximal n (or ∞) such that g_0 lifts to a homomorphism $X_i \otimes A/\mathfrak{m}^n \rightarrow Z_k \otimes A/\mathfrak{m}^n$. We recall [6, Lemma 3.6]:

Lemma 2.6. *Suppose that $f_0 \in \mathcal{O}_D \setminus \text{Hom}_{\mathcal{O}_F}(X_i, Z_k)$. Then*

$$n_{k+1}(\Pi f_0) = n_k(f_0) + e/e_{k+1}. \quad \square$$

Case $l < d$: Let us assume that $l < d$. Recall from Lemma 2.3 that if $l = 0$ and $i \neq j$, then $n_j(f_0) = e$. If $l \neq 0$, we can write $f_0 = \Pi^l g_0$ with g_0 of height 0. Then $n_{j-l}(g_0) = e \cdot e_j/e_{j-l} = e \cdot q^l$ and an inductive application of the previous lemma shows

$$n_j(f_0) = e \cdot (q^l + q^{l-1} + \dots + 1),$$

which proves Theorem 2.4 in the first case.

Remaining cases: Now assume that $l \geq d$ and write $f_0 = \Pi^d g_0$. Then

$$g_0 \in (\text{End}(X_i) + V_{l-d}) \setminus (\text{End}(X_i) + V_{l-d+1})$$

and by [11, Theorem 2.1], we have

$$n_i(g_0) = e \cdot e_j / e_i \begin{cases} a((l-d)/2) + a((l-d)/2 - 1) & \text{if } l-d < 2i \text{ is even} \\ 2a((l-d)/2) & \text{if } l-d < 2i \text{ is odd} \\ 2a(i-1) + \frac{1}{2}(l-(i+j-1))e_i & \text{if } l-d \geq 2i. \end{cases}$$

Again we apply Lemma 2.6 d times, distinguishing two cases. If $i = 0$, then only the fourth case of Theorem 2.4 and the third of the above cases occurs. The result is immediate. If $i > 0$, then one uses $e_j = q^d e_i$ to verify the formula. \square

3. ANALYTIC THEORY

In this section, we prove some general results about orbital integrals on $S(F)$. Our main results are Corollary 3.8 and Corollary 3.9 which characterize the functions of the form $\gamma \mapsto \text{Orb}_\gamma(f)$ and $\gamma \mapsto \partial \text{Orb}_\gamma(f)$. Another important result is Theorem 3.15 which plays a key role in the proof of the arithmetic transfer identity. Theorem 3.15 has been proven by W. Zhang for any n , see [15, Theorem 2.6].

We take up all notation from the introduction.

3.1. Orbital integrals on $S(F)$. First note that an element

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(E)$$

is regular semi-simple if and only if $b \neq 0$ and $c \neq 0$. We denote its entries by $a(\gamma), b(\gamma), c(\gamma)$ and $d(\gamma)$.

Parametrizations of $S(F)_{\text{rs}}$ and of $B := S(F) \setminus S(F)_{\text{rs}}$ are given as follows:

$$\begin{aligned} S(F)_{\text{rs}} &= \left\{ \begin{pmatrix} a & b \\ (1-N(a))/\bar{b} & -\bar{a}b/\bar{b} \end{pmatrix} \text{ with } b \neq 0 \text{ and } 1-N(a) \neq 0 \right\}, \\ B &= \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a, d \in U(1)(F) \text{ and } c\bar{a} + d\bar{c} = 0 \right\} \\ &\cup \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in U(1)(F) \text{ and } a\bar{b} + b\bar{d} = 0 \right\}. \end{aligned} \quad (3.1)$$

Denote by $B_0 \subset B$ the diagonal matrices, which we identify as $B_0 \cong U(1)(F) \times U(1)(F)$ by $\gamma \mapsto (a(\gamma), d(\gamma))$.

Remark 3.1. In general it is easy to understand locally constant functions on $S(F)_{\text{rs}}$ and their orbital integrals. The subtle point is to ensure that certain functions extend as locally constant functions to all of $S(F)$. The set B_0 will play a crucial role in this context. Namely any orbit passing close to $\gamma \in B$ will also pass close to B_0 . So any F^\times -invariant function f on $S(F)_{\text{rs}}$ is determined in a neighborhood of B by its behavior near B_0 .

Remark 3.2. Any function $f \in C_c^\infty(S(F))$ has an extension $\tilde{f} \in C_c^\infty(GL_2(E))$. It is also possible to define $\text{Orb}_\gamma(\tilde{f})$ for any regular semi-simple $\gamma \in GL_2(E)$. So we will often formulate topological statements for the group $GL_2(E)$, leaving the restriction to $S(F)$ implicit.

The integral $\text{Orb}_\gamma(f, s)$ transforms by the character $\eta_s(\cdot) := \eta(\cdot)|\cdot|^s$ under the action of F^\times . More precisely,

$$\text{Orb}_\gamma(\lambda^* f, s) = \eta_s^{-1}(\lambda) \text{Orb}_\gamma(f, s) = \text{Orb}_{\lambda^{-1}\gamma\lambda}(f, s).$$

Differentiating yields

$$\partial \text{Orb}_\gamma(\lambda^* f) = \eta(\lambda) [\partial \text{Orb}_\gamma(f) - \log |\lambda| \text{Orb}_\gamma(f)] = \partial \text{Orb}_{\lambda^{-1}\gamma\lambda}(f). \quad (3.2)$$

If χ is any character of F^\times , then we call a function ϕ on $S(F)_{\text{rs}}$ to be χ -invariant if $\phi(\lambda^{-1}\gamma\lambda) = \chi(\lambda)\phi(\gamma)$ for all $\lambda \in F^\times, \gamma \in S(F)_{\text{rs}}$. For example, $\gamma \mapsto \text{Orb}_\gamma(f, s)$ is η_s^{-1} -invariant for any test function $f \in C_c^\infty(S(F))$.

Lemma 3.3. *Let $f \in C_c^\infty(S(F))$.*

- a) $\gamma \mapsto \partial \text{Orb}_\gamma(f)$ is η -invariant if and only if all orbital integrals $\text{Orb}_\gamma(f)$ vanish.
- b) There exists $f' \in C_c^\infty(S(F))$ such that $\text{Orb}_\gamma(f) = \partial \text{Orb}_\gamma(f')$ for all $\gamma \in S(F)_{\text{rs}}$.

Proof. The first assertion follows directly from the transformation behavior of ∂Orb in (3.2). For the second assertion we compute

$$\partial \text{Orb}_\gamma(\eta(\lambda)f - \lambda^* f) = \eta(\lambda) \log |\lambda| \text{Orb}_\gamma(f).$$

Then we can choose $\lambda \in F^\times$ with $|\lambda| \neq 1$ and define $f' := (f - \eta(\lambda)\lambda^* f)/(\log |\lambda|)$. \square

From now on, we fix an extension of η to a smooth character on E^\times (not necessarily quadratic). It will again be denoted by η . This defines an extension of η_s , again denoted by η_s , to E^\times . Namely we set $\eta_s = \eta|\cdot|^s$ where $|\cdot|$ denotes the extension of the absolute value of F to E .

In light of Remark 3.1, the following proposition is very useful.

Proposition 3.4. *Given $f \in C_c^\infty(S(F))$, there exists $f' \in C_c^\infty(S(F))$ such that for all $\gamma \in S(F)_{\text{rs}}$,*

$$\begin{aligned} \text{Orb}_\gamma(f) &= \text{Orb}_\gamma(f'), \\ \partial \text{Orb}_\gamma(f) &= \partial \text{Orb}_\gamma(f') \end{aligned}$$

and $f'|_{B_0} = 0$.

Proof. By definition $f|_{B_0}$ is locally constant. Let

$$\mathcal{O}_E^\times \times \mathcal{O}_E^\times = \coprod V_a^i \times V_d^i$$

be a finite disjoint open covering such that $f|_{(V_a^i \times V_d^i) \cap B_0} \equiv r_i$ is constant.

For two open compact subsets $V_a, V_d \subset E$, let $1(V_a, V_d)$ be the characteristic function of the set

$$K(V_a, V_d) := \left\{ \begin{pmatrix} V_a & \mathcal{O}_E \\ \mathcal{O}_E & V_d \end{pmatrix} \right\} \cap S(F).$$

Note that this is a compact open subset of $S(F)$ and hence $1(V_a, V_d) \in C_c^\infty(S(F))$.

We choose $\lambda_0 \in F^\times$ with $\eta(\lambda_0) = -1$ and $v_F(\lambda_0) \geq 1$ to define

$$\alpha'(V_a, V_d) := 1(V_a, V_d) + \lambda_0^* 1(V_a, V_d).$$

By the η -invariance of $\text{Orb}_\gamma(f)$, we see that $\text{Orb}_\gamma(\alpha'(V_a, V_d)) = 0$ for all $\gamma \in S(F)_{\text{rs}}$.

Again we choose $\lambda_1 \in F^\times$ with $\eta(\lambda_1) = -1$ and $v_F(\lambda_1) \geq 1$ to define

$$\alpha(V_a, V_d) := \frac{1}{4} \left(\alpha'(V_a, V_d) + \lambda_1^* \alpha'(V_a, V_d) \right).$$

Then by Lemma 3.3 a), $\text{Orb}_\gamma(\alpha(V_a, V_d)) = \partial \text{Orb}_\gamma(\alpha(V_a, V_d)) = 0$ for all $\gamma \in S(F)_{\text{rs}}$.

It follows that

$$f' := f - \sum_i r_i \alpha(V_a^i, V_d^i)$$

satisfies the conditions of the proposition. \square

3.2. Germ expansion of orbital integrals. In the following, X denotes the space $\mathcal{O}_E^\times \times \mathcal{O}_E^\times \times (E^\times / F^\times)$. It contains a parameter space for the F^\times -orbits of each of the two components of B in (3.1).

Theorem 3.5. *Let $f \in C_c^\infty(S(F))$ such that $f|_{B_0} = 0$. Then there exist two locally constant functions*

$$A_0, A_1 : X \longrightarrow \mathbb{C}[q^s, q^{-s}]$$

such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(F)_{\text{rs}}$ near B_0 ,

$$\text{Orb}_\gamma(f, s) = \eta_s(b) A_0(s; a, d, b) + \eta_s(c)^{-1} A_1(s; a, d, c). \quad (3.3)$$

Conversely, given locally constant A_0, A_1 as above, there exists an $f \in C_c^\infty(S(F))$ such that identity (3.3) holds.

Definition 3.6. We call identity (3.3) the germ expansion of f . This terminology is taken from [14]. We also write $A_0(\gamma)$ and $A_1(\gamma)$ instead of $A_0(s; a, d, b)$ and $A_1(s; a, d, c)$.

Proof. Let $f \in C_c^\infty(S(F))$ with $f|_{B_0} = 0$.

Step 1: For $(a, d, b), (a, d, c) \in X$, we define

$$A_0(s; a, d, b) := \frac{1}{\eta_s(b)} \int_{F^\times} \eta_s(h) f\left(\begin{pmatrix} a & b/h \\ 0 & d \end{pmatrix}\right) dh,$$

$$A_1(s; a, d, c) := \eta_s(c) \int_{F^\times} \eta_s(h) f\left(\begin{pmatrix} a & 0 \\ ch & d \end{pmatrix}\right) dh.$$

Note that these integrals are absolutely convergent, since the integrand has no support near B_0 . They are F^\times -invariant in b and c and hence well defined.

Step 2: Now fix $(a, d) \in B_0$ and choose a neighborhood $V_a \times V_d \subset \mathcal{O}_E^\times \times \mathcal{O}_E^\times$ such that for all

$$\gamma \in \left\{ \begin{pmatrix} V_a & * \\ * & V_d \end{pmatrix} \right\},$$

the value $f(\gamma)$ is independent of $a(\gamma)$ and $d(\gamma)$. Such a neighborhood exists, since f is locally constant with compact support.

There exists an integer N and two finite families of elements c_i, b_j with $i = 1, \dots, n$ and $j = 1, \dots, m$ with the following properties. The open sets (in $S(F)$)

$$T_0 := \left\{ \begin{pmatrix} V_a & \pi^N \mathcal{O}_E \\ \pi^N \mathcal{O}_E & V_d \end{pmatrix} \right\}, \quad T_i := \left\{ \begin{pmatrix} V_a & \pi^N \mathcal{O}_E \\ c_i + \pi^N \mathcal{O}_E & V_d \end{pmatrix} \right\},$$

$$S_j := \left\{ \begin{pmatrix} V_a & b_j + \pi^N \mathcal{O}_E \\ \pi^N \mathcal{O}_E & V_d \end{pmatrix} \right\}$$

are disjoint, cover $\text{Supp } f \cap B$ and f is constant on each of them. Let Z be their union.

Then there exists a neighborhood W_0 of $\text{diag}(a, d)$ such that for any $\gamma \in W_0$, we have $F^\times \gamma \cap \text{Supp}(f) \subset Z$. Note that $f|_{T_0} \equiv 0$, so the maps $S(F)_{\text{rs}} \ni \gamma \mapsto \text{Vol}(F^\times \gamma \cap T_i)$ and $\gamma \mapsto \text{Vol}(F^\times \gamma \cap S_j)$ are locally constant in a neighborhood W_1 of $\text{diag}(a, d)$. Since they are clearly F^\times -invariant, they yield locally constant functions on X . It is now clear, that the theorem holds for $\gamma \in W_0 \cap W_1$. Since $(a, d) \in B_0$ was arbitrary, this finishes the proof of the first part of the theorem.

Proof of the second part: Now let A_0, A_1 be given. We want to construct a suitable function $f \in S(F)_{\text{rs}}$. By linearity and symmetry of the argument we assume that $A_1 = 0$ and that A_0 takes values in $q^{ks}\mathbb{C} \subset \mathbb{C}[q^s, q^{-s}]$ for some $k \in \mathbb{Z}$.

Reduction to $k = 0$: Assume for the moment the existence of f whenever A_0 takes values in $q^0\mathbb{C}$. Let f_0 be a function with germ expansion associated to $q^{-ks}A_0$ and $A_1 = 0$. Choose $\lambda_0 \in F^\times$ with valuation $v_F(\lambda_0) = k$. Then by the η_s^{-1} -invariance, $\eta(\lambda_0)\lambda_0^*f_0$ has germ expansion associated to A_0 and A_1 .

Case $k = 0$: So we can assume that A_0 takes values in \mathbb{C} . Let $K \subset E^\times$ be a compact open subset such that $F^\times \cdot K = E^\times$ and let β be the characteristic function of the set

$$\left\{ \begin{pmatrix} \mathcal{O}_E^\times & K \\ \mathcal{O}_E & \mathcal{O}_E^\times \end{pmatrix} \right\}.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(F)$ we define

$$f(\gamma) := \begin{cases} 0 & \text{if } F^\times b \cap K = \emptyset \\ \beta(\gamma)\eta_s(b) \text{Vol}(F^\times b \cap K)^{-1} A_0(a, d, b) & \text{otherwise.} \end{cases}$$

Clearly this function is locally constant with compact support. We compute the germ expansion of f as in the first part of the theorem. For this we assume that γ is near B_0 .

$$\begin{aligned} & \eta_s(b)^{-1} \int_{F^\times} \eta_s(h) f\left(\begin{pmatrix} a & b/h \\ 0 & d \end{pmatrix}\right) dh \\ &= \frac{\eta_s(b)^{-1}}{\text{Vol}(F^\times b \cap K)} \int_{F^\times b \cap K} \eta_s(h) \eta_s(b/h) A_0(a, d, b/h) dh \\ &= \frac{1}{\text{Vol}(F^\times b \cap K)} \int_{F^\times b \cap K} A_0(a, d, b) dh \\ &= A_0(a, d, b) \end{aligned}$$

In the second to last equality, we used that A_0 is invariant under multiplication with $h \in F^\times$. The integral A_1 vanishes since $0 \notin K$. This concludes the proof of the theorem. \square

Remark 3.7. The fact that f can locally near B_0 be defined by polynomials in q^s, q^{-s} which transform with η_s is equivalent to $f|_{B_0} = 0$. Namely if $f(\text{diag}(a, d)) \neq 0$, then the number of monomials q^{ks} in $\text{Orb}_\gamma(f, s)$ is not bounded for γ approaching $\text{diag}(a, d)$.

Corollary 3.8. *Let $f \in C_c^\infty(S(F))$ and fix $s \in \mathbb{C}$. Then the function (on $S(F)_{\text{rs}}$) $\phi : \gamma \mapsto \text{Orb}_\gamma(f, s)$ is η_s^{-1} -invariant and $\overline{\text{Supp}(\phi)}/F^\times$ is compact. There exist locally constant functions $A_0, A_1 : X \rightarrow \mathbb{C}$ such that for all regular semi-simple γ near B_0 ,*

$$\phi(\gamma) = \eta_s(b)A_0(\gamma) + \eta_s(c)^{-1}A_1(\gamma).$$

Conversely if $\phi \in C^\infty(S(F)_{\text{rs}})$ satisfies the above conditions, then there exists a function $f \in C_c^\infty(S(F))$ such that $\text{Orb}_\gamma(f, s) = \phi(\gamma)$ for all $\gamma \in S(F)_{\text{rs}}$.

Proof. Given f , we can assume that $f|_{B_0} = 0$ by similar arguments as in the proof of Proposition 3.4. Then we apply the first part of Theorem 3.5 and evaluate the functions A_0 and A_1 in the fixed value s .

Let now ϕ be given. By the second part of Theorem 3.5, there exists a function f_0 such that $\text{Orb}_\gamma(f_0, s) = \phi(\gamma)$ in a neighborhood of B_0 . By considering the difference $\text{Orb}_\gamma(f_0, s) - \phi(\gamma)$, we can assume that $\phi = 0$ in a neighborhood of B_0 . In particular $\text{Supp}(\phi) = \text{Supp}(\phi)$.

Let $K \subset S(F)_{\text{rs}}$ be open and compact, such that $F^\times K = \text{Supp}(\phi)$. Then $\mu(\gamma) := \text{Vol}(F^\times \gamma \cap K)$ is an F^\times -invariant smooth function on $S(F)_{\text{rs}}$ which vanishes in a

neighborhood of B . Note that $\phi(h^{-1}\gamma h) = h^*\phi(\gamma) = \eta_s(h)^{-1}\phi(\gamma)$ and set $f(\gamma) := 1_K(\gamma)\phi(\gamma)/\mu(\gamma)$. (If $\gamma \notin F^\times K$, then we define $f(\gamma) = 0$.) Then

$$\text{Orb}_\gamma(f, s) = \int_{F^\times} \eta_s(h) 1_K(h^{-1}\gamma h) \frac{\phi(h^{-1}\gamma h)}{\mu(h^{-1}\gamma h)} dh = \int_{F^\times \gamma \cap K} \frac{\phi(\gamma)}{\mu(\gamma)} dh = \phi(\gamma).$$

□

Let v be the extension of the normalized valuation from F to E .

Corollary 3.9. *Let $f \in C_c^\infty(S(F))$. Then there exist locally constant functions*

$$A_0, A'_0, A_1, A'_1 : X \longrightarrow \mathbb{C}$$

such that for all regular semi-simple $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ near B_0 , there is an identity

$$\partial \text{Orb}_\gamma(f) = \eta(b) \left[v(b) A_0(a, d, b) + A'_0(a, d, b) \right] + \eta(c)^{-1} \left[v(c) A_1(a, d, c) + A'_1(a, d, c) \right].$$

Conversely given A_0, A'_0, A_1, A'_1 as above, there exists a function $f \in C_c^\infty(S(F))$ such that $\gamma \mapsto \partial \text{Orb}_\gamma(f)$ satisfies the above identity near B_0 .

Proof. By Proposition 3.4, we can assume that $f|_{B_0} = 0$. Let

$$\text{Orb}_\gamma(f) = \eta_s(b) C_0(s; a, d, b) + \eta_s(c)^{-1} C_1(s; a, d, c)$$

be the germ expansion of f from Theorem 3.5. Its derivative in $s = 0$ is given as

$$\begin{aligned} \partial \text{Orb}_\gamma(f) &= \eta(b) \left(C'_0(0; a, d, b) - v(b) \log(q) C_0(0; a, d, b) \right) \\ &\quad + \eta(c)^{-1} \left(C'_1(0; a, d, c) + v(c) \log(q) C_1(0; a, d, c) \right). \end{aligned} \tag{3.4}$$

We now perform the obvious substitutions.

Conversely if A_0, A'_0, A_1, A'_1 are given, then we choose families of polynomials $C_0(s), C_1(s) : X \longrightarrow \mathbb{C}[q^s, q^{-s}]$ having the following values and derivatives in $s = 0$:

$$\begin{aligned} C'_0(0; \gamma) &= A'_0(\gamma) \text{ and } -\log(q) C_0(0; \gamma) = A_0(\gamma), \\ C'_1(0; \gamma) &= A'_1(\gamma) \text{ and } \log(q) C_1(0; \gamma) = A_1(\gamma). \end{aligned}$$

Then we apply the second part of Theorem 3.5. □

Corollary 3.10. *Let $f \in C_c^\infty(S(F))$ be such that $\text{Orb}_\gamma(f) = 0$ for all $\gamma \in S(F)_{\text{rs}}$. Then there exists a function $f' \in C_c^\infty(S(F))$ such that for all $\gamma \in S(F)_{\text{rs}}$,*

$$\partial \text{Orb}_\gamma(f) = \text{Orb}_\gamma(f').$$

Proof. By Corollary 3.8 it is enough to show that A_0 and A_1 in Corollary 3.9 vanish. Let us consider the germ expansion of f ,

$$\text{Orb}_\gamma(f) = \eta(b(\gamma)) C_0(\gamma) + \eta(c(\gamma))^{-1} C_1(\gamma).$$

It follows from formula (3.4), that $A_0 = -\log(q) C_0$ and $A_1 = \log(q) C_1$. Hence we need to show that $C_0 = C_1 = 0$.

Let $\lambda \in F^\times$ with $\eta(\lambda) = -1$. Then by the assumption on f , for any γ close enough to B_0 ,

$$\eta(b(\gamma)) C_0(\gamma) + \eta(c(\gamma))^{-1} C_1(\gamma) = 0$$

and

$$\eta(b(\gamma)) C_0(\gamma) + \eta(\lambda c(\gamma))^{-1} C_1(\lambda^{-1} \gamma \lambda) = 0.$$

Since $C_1(\lambda^{-1} \gamma \lambda) = C_1(\gamma)$, it follows that $C_0 = C_1 = 0$. □

Remark 3.11. Let $\chi : E^\times \rightarrow \mathbb{C}^\times$ be a smooth character or let χ_s be a family of such characters. Then all the statements in this subsection should have analogues for a χ -twisted orbital integral $\text{Orb}_\gamma(f, \chi)$ and the family $\text{Orb}_\gamma(f, \chi_s)$. They should also hold for orbital integrals on $GL_2(E)$, since we never used the structure of $S(F)$ in our proofs.

3.3. Transfer of Functions. For $\varepsilon \in F^\times$, we let $U^\varepsilon := U(\varepsilon \oplus 1)$ be the corresponding unitary group. Then $U^\varepsilon(F) \subset GL_2(E)$ is stable under the conjugation by $U(1)(F)$. For $\delta \in U^\varepsilon(F)_{\text{rs}}$ and $\varphi \in C_c^\infty(U^\varepsilon(F))$, we define

$$\text{Orb}_\delta(\varphi) = \int_{U(1)(F)} \varphi(h^{-1}\gamma h) dh.$$

Elements $\gamma \in S(F)_{\text{rs}}$ and $\delta \in U^\varepsilon(F)_{\text{rs}}$ are said to match, if they are conjugate under E^\times . A direct computation shows that a given γ matches some $\delta \in U^\varepsilon(F)$ if and only if $(1 - N(a))/\varepsilon$ is a norm of E/F . Assume this is the case and let $N(x) = (1 - N(a))/\varepsilon$. Then

$$\gamma \text{ matches } \delta = \begin{pmatrix} a & x \\ \bar{x}b\varepsilon/\bar{b} & -\bar{a}b/\bar{b} \end{pmatrix} \in U^\varepsilon(F). \quad (3.5)$$

Conversely, a given $\delta \in U^\varepsilon(F)_{\text{rs}}$ has a match in $S(F)$.

As in the introduction, we let $U_0 = U^{\varepsilon_0}$ and $U_1 = U^{\varepsilon_1}$ be the unitary groups associated to a norm ε_0 and a non-norm ε_1 . Then the matching relation defines a bijection of conjugation orbits:

$$[S(F)_{\text{rs}}] \cong [U_0(F)_{\text{rs}}] \sqcup [U_1(F)_{\text{rs}}].$$

For a given function $f \in C_c^\infty(S(F))$, the orbital integral $\gamma \mapsto \text{Orb}_\gamma(f)$ is not F^\times -invariant and hence cannot descend to the quotient $[S(F)_{\text{rs}}]$. This motivates the following definition.

Definition 3.12. We define the transfer factor ω by the formula

$$\omega(\gamma) := \eta(c(\gamma)). \quad (3.6)$$

This is a smooth η -invariant function on $S(F)_{\text{rs}}$.

Definition 3.13. We say that $f \in C_c^\infty(S(F))$ and $(\varphi_0, \varphi_1) \in C_c^\infty(U_0(F)) \times C_c^\infty(U_1(F))$ are transfers of each other if for each $\gamma \in S(F)_{\text{rs}}$ matching $\delta \in U_i(F)$, there is an equality

$$\omega(\gamma) \text{Orb}_\gamma(f) = \text{Orb}_\delta(g_i).$$

In particular, $f \in C_c^\infty(S(F))$ is a transfer of $(0, 0)$ if and only if $\text{Orb}_\gamma(f) = 0$ for all $\gamma \in S(F)_{\text{rs}}$.

Remark 3.14. Note that if $h \in E^\times$, then conjugation by h induces an isomorphism $U^\varepsilon \rightarrow U^{N(h)\varepsilon}$. If $\gamma \in S(F)_{\text{rs}}$ matches $\delta \in U^\varepsilon(F)$, then γ matches $h^{-1}\delta h \in U^{N(h)\varepsilon}(F)$. In particular, the pullback $h^* : C_c^\infty(U^{N(h)\varepsilon}(F)) \rightarrow C_c^\infty(U^\varepsilon(F))$ is an isomorphism which preserves the transfer in an obvious sense. It follows that the choice of ε_0 and ε_1 is irrelevant in this analytic setup.

The following theorem was already proven by W. Zhang in much greater generality, see [15, Theorem 2.6].

Theorem 3.15. *Given a pair of functions $(\varphi_0, \varphi_1) \in C_c^\infty(U_0(F)) \times C_c^\infty(U_1(F))$, there exists a transfer $f \in C_c^\infty(S(F))$.*

Conversely any function $f \in C_c^\infty(S(F))$ has a transfer (φ_0, φ_1) .

Proof. Consider a pair of functions $(\varphi_0, \varphi_1) \in C_c^\infty(U_0(F)) \times C_c^\infty(U_1(F))$ and the map

$$\phi : \gamma \mapsto \begin{cases} \omega(\gamma)^{-1} \text{Orb}_\delta(\varphi_0) & \text{if } \gamma \text{ matches } \delta \in U_0(F) \\ \omega(\gamma)^{-1} \text{Orb}_\delta(\varphi_1) & \text{if } \gamma \text{ matches } \delta \in U_1(F). \end{cases}$$

It is η -invariant and $\overline{\text{Supp}(\phi)}/F^\times$ is compact. According to Corollary 3.8 it is enough to show that ϕ has a germ expansion.

Let $i \in \{0, 1\}$ and fix a diagonal matrix $\text{diag}(a, d) \in U_i(F)$. The open neighborhoods $\text{diag}(a, d) + M_2(\pi^N \mathcal{O}_E)$ are stable under conjugation by $U(1)(F)$. Since φ_i is locally constant, it follows that $\text{Orb}_\delta(\varphi_i)$ is constant near $\text{diag}(a, d)$ with value $C_i(a, d) := \varphi_i(\text{diag}(a, d))$.

Note that each $\gamma \in S(F)_{\text{rs}}$ satisfies $c(\gamma) = (1 - N(a(\gamma)))/\overline{b(\gamma)}$ and $d(\gamma) = -\overline{a(\gamma)}b(\gamma)/\overline{b(\gamma)}$. Let b_0 be such that $-ab_0/\overline{b_0} = d$. Then if γ is close to $\text{diag}(a, d)$, then $b(\gamma)$ is close to $F^\times b_0$ and hence $\eta(b(\gamma)/\overline{b(\gamma)}) = \eta(b_0/\overline{b_0})$.

Expanding the definition of transfer, we see that we need to solve the following system of equations of functions on X :

$$\begin{aligned} & \eta(1 - N(a))\eta(b_0/\overline{b_0})A_0(a, d, b) + A_1(a, d, c) \\ &= \begin{cases} C_0(a, d) & \text{if } \eta(1 - N(a)) = 1 \\ C_1(a, d) & \text{if } \eta(1 - N(a)) = -1 \end{cases} \end{aligned}$$

A solution is given by

$$A_0(a, d, b) := \eta(\overline{b_0}/b_0)(C_0 - C_1)/2 \text{ and } A_1(a, d, c) := (C_0 + C_1)/2. \quad (3.7)$$

For the converse, let $f \in C_c^\infty(S(F))$ and fix $a, d \in U(1)(F)$. If a regular semi-simple γ is near $\text{diag}(a, d)$, then $\omega(\gamma)\eta(b(\gamma)) = \eta(1 - N(a))\eta(b_0/\overline{b_0}) = \pm\eta(b_0/\overline{b_0})$ as explained above. It follows that $\omega(\gamma) \text{Orb}_\gamma(f)$ takes only two values near $\text{diag}(a, d)$, depending on $\eta(1 - N(a))$. Denote these values by $C_0(a, d)$ and $C_1(a, d)$, meaning

$$\omega(\gamma) \text{Orb}_\gamma(f) = \begin{cases} C_0(a, d) & \text{if } 1 - N(a) \text{ is a norm} \\ C_1(a, d) & \text{if } 1 - N(a) \text{ is not a norm,} \end{cases}$$

whenever γ is near B_0 . Consider the map

$$\Psi_i : U_i(F)_{\text{rs}} \ni \delta \mapsto \omega(\gamma) \text{Orb}_\gamma(f)$$

where γ is a match for δ . We need to show that Ψ_0 and Ψ_1 are given by orbital integrals on $U_0(F)$ and $U_1(F)$.

Clearly there exist $(\varphi_0, \varphi_1) \in C_c^\infty(U_0(F)) \times C_c^\infty(U_1(F))$ with $\varphi_i(\text{diag}(a, d)) = C_i(a, d)$ for all $a, d \in U(1)(F)$. Then $\text{Orb}_\delta(\varphi_i) = C_i(a, d)$ for regular semi-simple δ in a neighborhood of $\text{diag}(a, d)$. We now consider the difference $\alpha_i(\delta) := \Psi_i(\delta) - \text{Orb}_\delta(\varphi_i)$. Denote its support by $K_i \subset U_i(F)_{\text{rs}}$. It is open and compact.

The function $\delta \mapsto \text{Vol}(U(1)(F)\delta \cap K_i)$ is locally constant and $U(1)(F)$ -invariant. For $\delta \in K_i$ we define

$$\tilde{\alpha}_i(\delta) := \begin{cases} 0 & \text{if } U(1)(F)\delta \cap K_i = \emptyset \\ \alpha_i(\delta) / \text{Vol}(U(1)(F)\delta \cap K_i) & \text{otherwise.} \end{cases}$$

Then $\text{Orb}_\delta(\tilde{\alpha}_i) = \alpha_i$. And hence $\Psi_i(\delta) = \text{Orb}_\delta(\varphi_i + \tilde{\alpha}_i)$. \square

4. THE ARITHMETIC TRANSFER IDENTITY

4.1. The group of quasi-isogenies G . In the introduction we defined $G \subset \text{Aut}^0(\mathbb{X}^{(2)})$ to be the group of \mathcal{O}_E -linear quasi-isogenies which preserve the polarization. We can identify this group as follows. Consider the embedding $E \longrightarrow M_2(D)$ defined by the action of \mathcal{O}_E on $\mathbb{X}^{(2)}$. It is given as

$$x \mapsto \begin{cases} \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix} & \text{if } E/F \text{ is ramified or if } i+j \text{ is even} \\ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} & \text{otherwise.} \end{cases}$$

In the first case, let $\varpi \in D$ be an element with $-\varpi^2 = \varepsilon_1$ and $\varpi a = \bar{a}\varpi$ for all $a \in \mathcal{O}_E$. Then the E -linear quasi-endomorphisms of $\mathbb{X}^{(2)}$ are given by $\varpi M_2(E)\varpi^{-1} \subset M_2(D)$ where the notation means conjugation by $\text{diag}(\varpi, 1)$. In the second case, the centralizer of E is $M_2(E) \subset M_2(D)$, and we let $\varpi \in E$ be such that $N(\varpi) = \varepsilon_0$. In any case, $G \subset \varpi M_2(E)\varpi^{-1}$ is the group of matrices A such that $A^*A = \text{id}_2$, where $*$ denotes the transposition and standard involution.

We now consider both G and $GL_2(E)$ as subgroups of $GL_2(D)$. Then conjugation with $\text{diag}(\varpi^{-1}, 1)$ defines an isomorphism

$$G \xrightarrow{\cong} \begin{cases} U_1(F) & \text{if } E/F \text{ is ramified or if } i+j \text{ is even} \\ U_0(F) & \text{otherwise.} \end{cases} \quad (4.1)$$

Let $S(F)_G \subset S(F)_{\text{rs}}$ be the elements which match in $U_1(F)$ if E/F is ramified or if $i+j$ is even. Otherwise let $S(F)_G$ be the elements matching in $U_0(F)$.

Definition 4.1. An element $\gamma \in S(F)_G$ is said to match $g \in G$, if it matches a $\delta \in U_i(F)$ which maps to g under isomorphism (4.1).

A direct computation using formula (3.5) shows that $\gamma \in S(F)_G$ matches

$$g = \begin{pmatrix} \varpi a \varpi^{-1} & \varpi x \\ -\bar{x} b \varepsilon / \bar{b} \cdot \varpi^{-1} & -\bar{a} b / \bar{b} \end{pmatrix}, \quad (4.2)$$

where $\varepsilon = \varepsilon_0$ or $\varepsilon = \varepsilon_1$ depending on the case and $N(x) = (1 - N(a))/\varepsilon$. We complement Remark 3.14 with the following corollary.

Corollary 4.2. *Let $\gamma \in S(F)_G$ match $g \in G$. Then g is unique up to conjugation by $U(1)(F)$, independent of the chosen $\varepsilon_0, \varepsilon_1$ in the definition of the unitary groups.*

Proof. This is immediate from the previous formula. Namely let $t \in D$ have Norm $1 - N(a)$. Then the product ϖx lies in $tU(1)(F)$, independently of ε . \square

4.2. The Arithmetic Fundamental Lemma. We recall the AFL in the case $n = 2$ in our terminology. This is done to illustrate similarities and differences to our arithmetic transfer identity as explained in the introduction. So only in this subsection, E/F is unramified, $i = j = 0$ and the extension of η to E^\times is given by $\eta(x) = (-1)^{v(x)}$.

Consider the formal \mathcal{O}_F -module $X_0 \times \bar{X}_0$ over $\text{Spf } \mathcal{O}_{\bar{E}}$ and let $g \in \text{End}^0(\mathbb{X} \times \bar{\mathbb{X}})$ be a quasi-endomorphism. We define $\text{Def}(g) \subset \text{Spf } \mathcal{O}_{\bar{E}}$ to be the maximal closed subscheme to which g deforms as endomorphism. We set $\text{Int}(g) := \text{len}_{\mathcal{O}_{\bar{E}}} \text{Def}(g)$, possibly ∞ . Then the Arithmetic Fundamental lemma for $n = 2$ is the following theorem.

Theorem 4.3. [13, Theorem 2.10] *Let E/F be unramified and assume that in the definition of the unitary groups $\varepsilon_0 = 1$ and $\varepsilon_1 = \pi_F$. Assume that $\gamma \in S(F)_G$ matches g in G . Let 1_K be the characteristic function of $K = GL_2(\mathcal{O}_E) \cap S(F)$. Then $\text{Def}(g)$ is artinian and there is an equality*

$$\omega(\gamma) \partial \text{Orb}_\gamma(1_K) = \text{Int}(g) \cdot \log q. \quad (4.3)$$

Furthermore, the function 1_K has transfer $(1_{U_0(\mathcal{O}_E)}, 0)$.

Proof. Recall that γ has the form

$$\gamma = \begin{pmatrix} a & b \\ (1 - N(a))/\bar{b} & -\bar{a}b/\bar{b} \end{pmatrix}.$$

The condition that γ matches in $U_1(F)$ is equivalent to $\pi_F(1 - N(a))$ being a norm and hence to $v(1 - N(a))$ being odd. In particular, $a \in \mathcal{O}_E$. We use formula (4.2) with $\varepsilon = \pi_F$ to compute g .

Calculation of the right hand side: We can write g as a matrix

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$$

with all $g_i \in \text{End}^0(\mathbb{X})$. Deforming g is equivalent to deforming all four entries separately since $\mathbb{X} \times \bar{\mathbb{X}}$ is lifted factor wise. The entries g_2 and g_3 Galois-commute with the \mathcal{O}_E -action. So by Corollary 2.2 b), they do not lift to $\text{End}(X_0)$. It follows that $\text{Def}(g)$ is artinian.

The entries g_1 and g_4 lie in \mathcal{O}_E and deform arbitrarily far. The entries g_2, g_3 both have valuation equal to

$$v_D(\varpi x) = 1 + v_D(x) = 1 + v_F((1 - N(a))/\pi_F) = v_F(1 - N(a)).$$

By Theorem 2.4, the length of the deformation locus equals $\frac{1}{2}(1 + v_F(1 - N(a)))$.

For the analytic side, we compute

$$\begin{aligned} \partial \text{Orb}_\gamma(1_K) &= -\log(q) \int_{F^\times} v(h) \eta(h) 1_K \left(\begin{pmatrix} a & b/h \\ h(1 - N(a))/\bar{b} & -\bar{a}b/\bar{b} \end{pmatrix} \right) dh \\ &= \log(q) \sum_{i=v(b)-v(1-N(a))}^{v(b)} (-1)^{i+1} i \end{aligned}$$

Now we use that $v(1 - N(a))$ is odd, so that the sum has an even number of summands to see

$$\begin{aligned} \partial \text{Orb}_\gamma(1_K) &= (-1)^{v(b)-v(1-N(a))} \log(q) \sum_{i=0}^{v(1-N(a))} (-1)^{i+1} i \\ &= \omega(\gamma)^{-1} \log(q) \frac{1 + v(1 - N(a))}{2}. \end{aligned}$$

□

4.3. The Arithmetic Transfer Identity. Now we fix two quasi-canonical lifts X_i and Y_j of levels i and j , defined over a finite extension $A/\mathcal{O}_{\bar{F}}$ of ramification index e . The formal \mathcal{O}_F -module $X_i \times Y_j$ is then a deformation of $\mathbb{X}^{(2)}$. For a given quasi-homomorphism $g \in \text{End}^0(\mathbb{X}^{(2)})$, we define $\text{Def}(g)$ to be the maximal closed subscheme of $\text{Spf } A$ to which g deforms as endomorphism of $X_i \times Y_j$. Let $\text{Int}(g) := \text{len}_{\mathcal{O}_{\bar{E}}} \text{Def}(g)$ as above.

Recall that $K_{i,j}$ denotes the stabilizer of $\mathcal{O}_i \oplus \mathcal{O}_j$ in $U_k(F)$, where $k = 0$ if E/F is ramified or if $i + j$ is even and $k = 1$ otherwise. Let $1_{K_{i,j}}$ be its characteristic function. The following theorems are our main results. Note that Theorem 4.4 follows for E/F unramified and $i = j = 0$ from the AFL (Theorem 4.3).

Theorem 4.4. *There exists a function $f \in C_c^\infty(S(F))$ which is a transfer of*

$$\begin{cases} (e \cdot 1_{K_{i,j}}, 0) & \text{if } E/F \text{ ramified or if } i + j \text{ is even} \\ (0, e \cdot 1_{K_{i,j}}) & \text{if } E/F \text{ unramified and if } i + j \text{ is odd} \end{cases}$$

with the following property. For any $\gamma \in S(F)_G$ matching $g \in G$, the length $\text{Int}(g)$ is finite and there is an equality

$$\omega(\gamma) \partial \text{Orb}_\gamma(f) = \text{Int}(g) \cdot \log(q). \quad (4.4)$$

Theorem 4.5. *For every function $f \in C_c^\infty(S(F))$ which is a transfer of*

$$\begin{cases} (e \cdot 1_{K_{i,j}}, 0) & \text{if } E/F \text{ ramified or if } i + j \text{ is even} \\ (0, e \cdot 1_{K_{i,j}}) & \text{if } E/F \text{ unramified and if } i + j \text{ is odd,} \end{cases}$$

there exists a function $f_{\text{corr}} \in C_c^\infty(S(F))$ such that for any $\gamma \in S(F)_G$ matching $g \in G$:

$$\omega(\gamma) \partial \text{Orb}_\gamma(f) = \text{Int}(g) \cdot \log(q) + \omega(\gamma) \text{Orb}_\gamma(f_{\text{corr}}). \quad (4.5)$$

Proof of Theorem 4.4. Let us assume Theorem 4.5. By Theorem 3.15, there exists a function f_0 which has the correct transfer. Let f_{corr} yield the correction term in Theorem 4.5. By Lemma 3.3 b), there exists a function f_1 with $\partial \text{Orb}_\gamma(f_1) = \text{Orb}_\gamma(f_{\text{corr}})$ for all $\gamma \in S(F)_{\text{rs}}$. By Lemma 3.3 a), f_1 has transfer $(0, 0)$. It follows that $f := f_0 + f_1$ is a function as in Theorem 4.4. \square

Proof of Theorem 4.5. Let f be a function as in the theorem. Denote by β the characteristic function of the set

$$\mathcal{O}_i^\times \times \mathcal{O}_j^\times \times (E^\times / F^\times) \subset X.$$

By formula (3.7) in the proof of Theorem 3.15, the germ expansion of f is given by $A_0 = \pm \frac{1}{2} \eta(\bar{b}/b) e \cdot \beta$ and $A_1 = \frac{1}{2} e \cdot \beta$. The sign is $+$ precisely if E/F is ramified or if $i + j$ is even.

By the formula in Corollary 3.9, this determines the leading terms in the germ expansion of $\partial \text{Orb}_\gamma(f)$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be near B_0 . Then

$$\partial \text{Orb}_\gamma(f) = \frac{e \log(q)}{2} \left(\mp \eta(\bar{b}) v(b) + \eta(c)^{-1} v(c) \right) \beta(\gamma) + R(\gamma),$$

where $R(\gamma)$ is constant part of the germ expansion. The sign change comes from the derivative of $\eta_s(b) = q^{-sv(b)} \eta(b)$. We now use that $c = (1 - N(a))/\bar{b}$ and (3.6) to get

$$\omega(\gamma) \partial \text{Orb}_\gamma(f) = \frac{e \log(q)}{2} \cdot \left(\mp \eta(1 - N(a)) v(b) + v(c) \right) \beta(\gamma) + \omega(\gamma) R(\gamma).$$

Finally if we choose $\gamma \in S(F)_G$, then $\eta(1 - N(a)) = -1$ precisely in the first case of the theorem. So for $\gamma \in S(F)_G$ near B_0 , the left hand side of (4.5) equals

$$\omega(\gamma) \partial \text{Orb}_\gamma(f) = \frac{e \log(q)}{2} v(1 - N(a)) \beta(\gamma) + \omega(\gamma) R(\gamma).$$

Now let us turn to the geometric side. Let $g \in G$ be a match for some $\gamma \in S(F)_G$. We write g as a matrix with entries g_1, g_2, g_3, g_4 . With the same arguments as in the proof of Theorem 4.3, $\text{Int}(g)$ is finite. We consider the difference $o(\gamma) := \omega(\gamma) \partial \text{Orb}_\gamma(f) - \text{Int}(g) \log(q)$ for $\gamma \in S(F)_G$.

Claim: The function $o(\gamma)$ (on $S(F)_G$) is constant near B_0 .

First note that $v_D(g_2)$ and $v_D(g_3)$ will tend to infinity as γ approaches B_0 . This follows from formula (4.2). Now fix $\text{diag}(a, d) \in B_0$. We distinguish two cases.

If $(a, d) \notin \mathcal{O}_i^\times \times \mathcal{O}_j^\times$, then g_1 and g_4 will not deform arbitrarily far for g near $\text{diag}(a, d)$. In fact, the length of the maximal subscheme of $\text{Spf } A$ to which they deform is computed in Theorem 2.4. This length equals $\text{Int}(g)$ for all γ near $\text{diag}(a, d)$.

If instead $(a, d) \in \mathcal{O}_i^\times \times \mathcal{O}_j^\times$, then g_1 and g_4 deform arbitrarily far for all γ near $\text{diag}(a, d)$. By Theorem 2.4, $\text{Int}(g)$ grows linearly in $v(1 - N(a))$ for γ approaching $\text{diag}(a, d)$. More precisely, $\text{Int}(g) - \frac{e}{2}v(1 - N(a))$ is constant near $\text{diag}(a, d)$. It follows that $o(\gamma)$ is constant near $\text{diag}(a, d)$, which proves the claim.

We define $A_0(a, d)$ to be the value of $o(\gamma)$ near $\text{diag}(a, d)$ and extend it to a smooth function on X . By definition, $\omega^{-1}o$ has a germ expansion associated to A_0 and $A_1 := 0$ when restricted to $\gamma \in S(F)_G$, i.e.

$$\omega(\gamma)^{-1}o(\gamma) = \eta(c(\gamma))^{-1}A_0(a(\gamma), d(\gamma)).$$

By the second part of Corollary 3.8, there exists a function f_{corr} satisfying

$$\omega(\gamma) \text{Orb}_\gamma(f_{\text{corr}}) = o(\gamma) \quad \forall \gamma \in S(F)_G.$$

This finishes the proof of Theorem 4.5. \square

We conclude with a result about the naturality of Theorems 4.4 and 4.5.

Proposition 4.6. *The function $e \cdot 1_{K_{i,j}}$ in the theorems can be replaced by any function g_0 such that $g_0 - e \cdot 1_{K_{i,j}}$ vanishes in a neighborhood of $U_k(F) \setminus U_k(F)_{\text{rs}}$.*

Proof. This is clear, since the proof of Theorem 4.5 only depended on $e \cdot 1_{K_{i,j}}|_{B_0}$. \square

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